Lesson 16 Linear Algebraic Equation

Exercise no. 1 Navie Gauss Elimination with out Pivoting

Use of Gauss Elimination without pivoting to solve the system of linear algebraic equation as given below

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \qquad (1)$$
$$0.1x_1 - 7x_2 - 0.3x_3 = -19.3 \qquad (2)$$
$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4 \qquad (3)$$

The first part of the procedure is forward elimination. Multiply Eq. (1) by 0.1/3 and subtract the result from Eq. (2) to give

$$7.00333x_2 - 0.293333x_3 = -19.5617$$

Then multiply Eq. (1) by 0.3/3 and subtract it from Eq. (3). After this the set of equation is

$$3x_1 - \quad 0.1x_2 - 0.2x_3 = 785 \qquad (4)$$
$$7.00333x_2 - 0.293333x_3 = -19.5617 \qquad (5)$$
$$-0.190000x_2 + 10.0200x_3 = 70.6150 \qquad (6)$$

To complete the forward elimination, x_2 must be removed from Eq. (6). To accomplish this, multiply Eq. (5) by 0.190000/7.00333 and subtract the result from Eq. (6). This eliminates x_2 from the third equation and reduces the system to an upper triangular form, as in

$$3x_1 - \quad 0.1x_2 - 0.2x_3 = 785 \qquad (7)$$
$$7.00333x_2 - 0.293333x_3 = -19.5617 \qquad (8)$$
$$10.0120x_3 = 70.0843 \qquad (9)$$

We can now solve these equations by back substitution. First, Eq. (9) can be solved for

$$x_3 = \frac{70.0843}{10.0120} = 7.00003$$

This result can be back-substituted into Eq. (8), which can then be solved for

$$x_3 = \frac{-19.5617 + 0.293333(7.00003)}{7.00003} = -2.500700$$

Finally, x_3 = 7.00003 and x_2 = -2.50000 can be substituted back into Eq. (7), which can be solved for

$$x_1 = \frac{7.85 + 0.1(-2.50000) + 0.2(7.00003)}{3} = 3.00000$$

Although there is a slight round-off error, the results are very close to the exact solution of $x_1 = 3$, $x_2 = -2.5$, and $x_3 = 7$. This can be verified by substituting the results into the original equation set:

$$\begin{aligned}
&3(3) - 0.1(-2.50000) + 0.2(7.00003) &&= 7.84999 \cong 7.85 \\
&0.1(3) + 7(-2.5) - 0.3(7.00003) &&= -19.30000 \ \cong 19.3 \\
&0.3(3) - 0.2(-2.5) + 10(7.00003) &&= 71.4003 \cong 71.4
\end{aligned}$$

The following MATLAB CODE will solve the system of linear equation as discussed above using Guass elimination without pivoting.

```
function x = GuassNaive(A,b)
%GuassNaive: naive Guass elimination
%    x = GuassNaive(A,b):Guass elimination without pivoting.
% input:
%    A = coefficient matrix
%    b = right hand side vector
% Output:
[m,n]=size(A);
if m~=n, error('Matrix A must be square'); end
nb = n+1;
Aug = [A b];
% forward elimination
for k = 1:n-1
    for i = k+1:n
        factor = Aug(i,k)/Aug(k,k);
        Aug(i,k:nb)=Aug(i,k:nb)-factor*Aug(k,k:nb);
    end
end
% back substitution
 x = zeros(n,1);
 x(n)=Aug(n,nb)/Aug(n,n);
 for i = n-1:-1:1
     x(i)=(Aug(i,nb)-Aug(i,i+1:n)*x(i+1:n))/Aug(i,i);
 end
```

Notice that the coefficient of matrix `A` and the right-hand-side vector `b` are combined in the augmented matrix `Aug`. Thus, the operations are performed on `Aug` rather than separately on `A` and `b`. Two nested loops provide a concise representation of the forward elimination step. The outer loop moves down the matrix from one pivot row to the next. The inner loop moves below the below the pivot row to each of the subsequent rows where elimination is to take place. Finally, the actual elimination is represented by a single line that takes advantage of MATLAB's ability to perform matrix operations.

Solution:

Go to command window and do the following operations

```
>> A=[3 -0.1 -0.2; 0.1 7 -0.3; 0.3 -0.2 10]


A =
    3.0000   -0.1000   -0.2000
    0.1000    7.0000   -0.3000
    0.3000   -0.2000   10.0000


>> b =[7.85 -19.3 71.4]'
b =
    7.8500
  -19.3000
   71.4000
>> GuassNaive(A,b)
ans =
    3.0000
   -2.5000
    7.0000
```

Exercise no. 2 Guass Jordan Method

This method is a modification of the Gaussian elimination method. The Gauss-Jordan method, however, is inefficient for practical calculation, but is often useful for theoretical purposes. The basis of this method is to convert the given matrix into diagonal form. The forward elimination of the Gauss-Jordan method is identical to that of the Gaussian elimination method. However, Gauss-Jordan elimination uses backward elimination rather than backward substitution. In the Gauss-Jordan method the forward elimination and backward elimination need not be separated. This is possible because a pivot element can be used to eliminate the coefficients not only below but also above at the same time. If this approach is taken, the form of the coefficients matrix becomes diagonal when elimination by the last pivot is completed. The Gauss-Jordan method simply yields a transformation of the augmented matrix of the form

$$[A|\mathbf{b}] \to [\mathbf{I}|\mathbf{c}],$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{c}$ is the column matrix, which represents the possible solution of the given linear system.

Solve the following linear system using the Gauss-Jordan method

$$
\begin{array}{rcrcrcr}
x_1 & + & 2x_2 & & & = & 3 \\
-x_1 & & & - & 2x_3 & = & -5 \\
-3x_1 & - & 5x_2 & + & x_3 & = & -4.
\end{array}
$$

Solution. *Write the given system in augmented matrix form*

$$
\left(\begin{array}{rrr:r} 1 & 2 & 0 & 3 \\ -1 & 0 & -2 & -5 \\ -3 & -5 & 1 & -4 \end{array}\right).
$$

The first elimination step is to eliminate the elements $a_{21} = -1$ and $a_{31} = -3$ by subtracting the multiples $m_{21} = -1$ and $m_{31} = -3$ of row 1 from rows 2 and 3, respectively, which gives

$$
\left(\begin{array}{rrr:r} 1 & 2 & 0 & 3 \\ 0 & 2 & -2 & -2 \\ 0 & 1 & 1 & 5 \end{array}\right).
$$

The second row is now divided by 2 to give

$$
\left(\begin{array}{rrr:r} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & 1 & 5 \end{array}\right).
$$

The second elimination step is to eliminate the elements in positions $a_{12}^{(1)} = 2$ and $a_{32} = 1$ by subtracting the multiples $m_{12} = 2$ and $m_{32} = 1$ of row 2 from rows 1 and 3, respectively, which gives

$$
\left(\begin{array}{rrr:r} 1 & 0 & 2 & 5 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & 6 \end{array}\right).
$$

The third row is now divided by 2 to give

$$
\left(\begin{array}{rrr:r} 1 & 0 & 2 & 5 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right).
$$

The third elimination step is to eliminate the elements in positions $a_{23}^{(1)} = -1$ and $a_{13} = 2$ by subtracting the multiples $m_{23} = -1$ and $m_{13} = 2$ of row 3 from rows 2 and 1, respectively, which gives

$$
\left(\begin{array}{rrr:r} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right).
$$

Obviously, the original set of equations has been transformed to a diagonal form. Now expressing the set in algebraic form yields

$$\begin{aligned} x_1 &= -1 \\ x_2 &= 2 \\ x_3 &= 3, \end{aligned}$$

which is the required solution of the given system.

Now we can get the above results by creating a function file and executing it in command window of MATLAB. To do so follow the MATLAB Code given below

```
function sol = GaussJ(Ab)
[m,n]=size(Ab);
for i = 1:m
    Ab(i,:)=Ab(i,:)/Ab(i,i);
    for j = 1:m
        if j == i;
            continue; end
        Ab(j,:)=Ab(j,:)-Ab(j,i)*Ab(i,:);
    end;
end;
sol = Ab
```

Save the above file in the name of `GaussJ.m` and follow the procedure given below:

```
>> Ab=[1 2 0 3; -1 0 -2 -5; -3 -5 1 -4]
Ab =

     1     2     0     3
    -1     0    -2    -5
    -3    -5     1    -4


>> GaussJ(Ab)
sol =

     1     0     0    -1
     0     1     0     2
     0     0     1     3

ans =

     1     0     0    -1
     0     1     0     2
     0     0     1     3
```