Part II Numerical Analysis

Lesson 10 Finding roots of a polynomial

In MATLAB, a polynomial is expressed as a row vector of the form [a_n a_{n-1} a_2 a_1 a_0]. The elements a_i of this vector are the coefficients of the polynomial in descending order. We must include terms whose coefficients are zero. We can find the roots of any polynomial with the **roots (p)** function where **p** is a row vector containing the polynomial coefficients in descending order.

Exercise 1. Find the roots of the polynomial

$$p_2(x) = x^5 - 7x^4 + 16x^2 + 25x + 52$$

There is no cube term; therefore, we must enter zero as its coefficient. The roots are found with the statements below where we have defined the polynomial as p2, and the roots of this polynomial as roots_ p2. The result indicates that this polynomial has three real roots, and two complex roots.

```
p2=[1 –7 0 16 25 52]
p2 =
    1   -7   0   16   25   52
roots_ p2=roots(p2)
roots_p2 =
        6.5014
        2.7428
       -1.5711
       -0.3366 + 1.3202i
       -0.3366 - 1.3202i
```

Exercise 2. Find the roots of the polynomial

$$p_1(x) = x^4 - 10\,x^3 + 35\,x^2 - 50x + 24$$

The roots are found with the following two statements. We have denoted the polynomial as p1, and the roots as roots_ p1.

```
p1=[1 –10 35 –50 24] % Specify the coefficients of p1(x)
p1 =
    1 -10 35 -50 24
roots_ p1=roots(p1) % Find the roots of p1(x)
roots_p1 =
    4.0000
    3.0000
    2.0000
    1.0000
```

From the above results we observe that MATLAB displays the polynomial coefficients as a row vector, and the roots as a column vector.

Exercise 3. Solving Quadratic Equations

You can solve equations involving variables with **`solve`** or **`fzero`**. To find the solutions of the quadratic equation,

$$x^2 - 2x - 4 = 0$$

type the following in command prompt

```
>> solve('x^2 - 2*x - 4 = 0')

ans =
    [ 5^(1/2)+1]
    [ 1-5^(1/2)]
```

Note that the equation to be solved is specified as a string; that is, it is surrounded by single quotes. The answer consists of the exact (symbolic) solutions $1 \pm \sqrt{5}$.

LESSON 11 **Matrices**

Exercise 1. Compute *A*+*B* and *A*–*B* given that

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 3 & 0 \\ -1 & 2 & 5 \end{bmatrix}$$

Solution:

$$A + B = \begin{bmatrix} 1+2 & 2+3 & 3+0 \\ 0-1 & 1+2 & 4+5 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 3 \\ -1 & 3 & 9 \end{bmatrix}$$

and

$$A - B = \begin{bmatrix} 1-2 & 2-3 & 3-0 \\ 0+1 & 1-2 & 4-5 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

Check with MATLAB in command prompt:

```
A=[1  2  3;   0  1  4];  B=[2  3  0; –1  2  5];     % Define matrices A and B
A+B                                                % Add A and B

ans =
      3       5       3
     -1       3       9

A–B                                                % Subtract B from A

ans =
     -1      -1       3
      1      -1      -1
```

Exercise 2. Transpose of a matrix

The transpose of a matrix *A*, denoted as A^T, is the matrix that is obtained when the rows and columns of matrix *A* are interchanged. For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ then } A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

In MATLAB we use the apostrophe (′) symbol to denote and obtain the transpose of a matrix. Thus, for the above example,

```
A=[1  2  3;  4  5  6] % Define matrix A
```

```
A =
     1     2     3
     4     5     6
```

```
A' % Display the transpose of A
```

```
ans =
     1     4
     2     5
     3     6
```

Exercise 3. (a) Determinants

Given that

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$$

compute $detA$ and $detB$.

Solution:

$$detA = 1 \cdot 4 - 3 \cdot 2 = 4 - 6 = -2$$
$$detA = 2 \cdot 0 - 2 \cdot (-1) = 0 - (-2) = 2$$

Check with MATLAB:

```
A=[1  2; 3  4]; B=[2  -1; 2  0];      % Define matrices A and B
det(A)                                % Compute the determinant of A
```

```
ans =
    -2
```

```
det(B)                                % Compute the determinant of B
```

```
ans =
     2
```

Exercise 3. (b) Compute the determinant of A using the elements of the first row.

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -4 & 2 \\ -1 & 2 & -6 \end{bmatrix}$$

Solution:

$$detA = 1\begin{bmatrix} -4 & 2 \\ 2 & -6 \end{bmatrix} - 2\begin{bmatrix} 2 & 2 \\ -1 & -6 \end{bmatrix} - 3\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} = 1 \times 20 - 2 \times (-10) - 3 \times 0 = 40$$

Check with MATLAB:

```
A=[1  2  –3; 2  –4  2; –1  2  –6];  det(A)      % Define matrix A and compute detA
ans =
    40
```

Exercise 4. Cramer's Rule

Use Cramer's rule to find v_1, v_2, and v_3 if

$$\begin{aligned} 2v_1 - 5 - v_2 + 3v_3 &= 0 \\ -2v_3 - 3v_2 - 4v_1 &= 8 \\ v_2 + 3v_1 - 4 - v_3 &= 0 \end{aligned}$$

and verify your answers with MATLAB.

Solution:

Rearranging the unknowns v, and transferring known values to the right side, we get

$$\begin{aligned} 2v_1 - v_2 + 3v_3 &= 5 \\ -4v_1 - 3v_2 - 2v_3 &= 8 \\ 3v_1 + v_2 - v_3 &= 4 \end{aligned}$$

Now, by Cramer's rule,

$$\Delta = \left|\begin{array}{ccc|cc} 2 & -1 & 3 & 2 & -1 \\ -4 & -3 & -2 & -4 & -3 \\ 3 & 1 & -1 & 3 & 1 \end{array}\right. = 6+6-12+27+4+4 = 35$$

$$D_1 = \left|\begin{array}{ccc|cc} 5 & -1 & 3 & 5 & -1 \\ 8 & -3 & -2 & 8 & -3 \\ 4 & 1 & -1 & 4 & 1 \end{array}\right. = 15+8+24+36+10-8 = 85$$

$$D_2 = \left|\begin{array}{ccc|cc} 2 & 5 & 3 & 2 & 5 \\ -4 & 8 & -2 & -4 & 8 \\ 3 & 4 & -1 & 3 & 4 \end{array}\right. = -16-30-48-72+16-20 = -170$$

$$D_3 = \left|\begin{array}{ccc|cc} 2 & -1 & 5 & 2 & -1 \\ -4 & -3 & 8 & -4 & -3 \\ 3 & 1 & 4 & 3 & 1 \end{array}\right. = -24-24-20+45-16-16 = -55$$

Therefore, using (4.31) we get

$$x_1 = \frac{D_1}{\Delta} = \frac{85}{35} = \frac{17}{7} \qquad x_2 = \frac{D_2}{\Delta} = -\frac{170}{35} = -\frac{34}{7} \qquad x_3 = \frac{D_3}{\Delta} = -\frac{55}{35} = -\frac{11}{7}$$

We will verify with MATLAB as follows

```
% The following code will compute and display the values of v1, v2 and v3.
format rat                            % Express answers in ratio form
B=[2 −1 3; −4 −3 −2; 3 1 −1];         % The elements of the determinant D
delta=det(B);                         % Compute the determinant D of B

detd1=det(d1);                        % Compute the determinant of D1
d2=[2 5 3; −4 8 −2; 3 4 −1];          % The elements of D2
detd2=det(d2);                        % Compute the determinant of D2
d3=[2 −1 5; −4 −3 8; 3 1 4];          % The elements of D3
detd3=det(d3);                        % Compute he determinant of D3
v1=detd1/delta;                       % Compute the value of v1
v2=detd2/delta;                       % Compute the value of v2
v3=detd3/delta;                       % Compute the value of v3
%
disp('v1=');disp(v1);                 % Display the value of v1
disp('v2=');disp(v2);                 % Display the value of v2
disp('v3=');disp(v3);                 % Display the value of v3
```

```
v1=
     17/7
v2=
    -34/7
v3=
    -11/7
```

Exercise 5. Solution of Simultaneous Equations with Matrices

Consider the relation,

$AX = B$

where A and B are matrices whose elements are known, and X is a matrix (a column vector) $AX = B$ whose elements are the unknowns. We assume that A and X are conformable for multiplication. Multiplication of both sides of $AX = B$ by A^{-1} yields:

$$A^{-1}AX = A^{-1}B = IX = A^{-1}B$$

or

$$\boxed{X = A^{-1}B}$$

Therefore, we can use the above equation to solve any set of simultaneous equations that have solutions. We will refer to this method as the inverse matrix method of solution of simultaneous equations.

Given the system of equations

$$\begin{Bmatrix} 2x_1 + 3x_2 + x_3 = 9 \\ x_1 + 2x_2 + 3x_3 = 6 \\ 3x_1 + x_2 + 2x_3 = 8 \end{Bmatrix}$$

Compute the unknown's x_1, x_2, and x_3 using the inverse matrix method.

Solution:

In matrix form, the given set of equations is $AX = B$ where

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

Then,

$$X = A^{-1}B$$

or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

Next, we find the determinant $detA$, and the adjoint $adjA$.

$$detA = 18 \quad and \quad adjA = \begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}$$

Therefore,

$$A^{-1} = \frac{1}{detA} adjA = \frac{1}{18}\begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}$$

and by $X = A^{-1}B$ we obtain

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}\begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 35 \\ 29 \\ 5 \end{bmatrix} = \begin{bmatrix} 35/18 \\ 29/18 \\ 5/18 \end{bmatrix} = \begin{bmatrix} 1.94 \\ 1.61 \\ 0.28 \end{bmatrix}$$

To verify our results, we could use the MATLAB inv(A) function, and multiply A^{-1} by B. However, it is easier to use the *matrix left division* operation $X = A \backslash B$; this is MATLAB's solution of $A^{-1}B$ for the matrix equation $A \cdot X = B$, where matrix X is the same size as matrix B. For this example,

```
A=[2  3  1; 1  2  3; 3  1  2]; B=[9  6  8]'; X=A \ B    % Observe that B is a column vector
```

```
X =
    1.9444
    1.6111
    0.2778
```

Exercise 6. Finding Eigen values and Eigen vectors

1. Find eigen vales and eigen vectors of the following 3 x 3 matrix using command window

```
>> A=[5 -3 2; -3 8 4; 4 2 -9];
>> eig(A)

ans =

  -10.2206
    4.4246
    9.7960

>> [eigvec,eigval]=eig(A)

eigvec =

    0.1725    0.8706   -0.5375
    0.2382    0.3774    0.8429
   -0.9558    0.3156   -0.0247

eigval =

  -10.2206         0         0
         0    4.4246         0
         0         0    9.7960
```

2. Find the eigen value and eigen vector for the following 2 X 2 matrix using command window

```
>> A=[-4.0 4.0; -1.6 1.2];
>> [eigvec,eigval]=eig(A)
eigvec =

   -0.8944   -0.7809
   -0.4472   -0.6247
eigval =

   -2.0000         0
         0   -0.8000
```

Lesson 12 Ordinary Differential Equations

Exercise 1. Solving a first order ordinary differential equation

Solve the first–order linear differential equation with the initial conditions $x(0) = 0$.

Step 1: Write the equation(s) as a system of first-order equations:

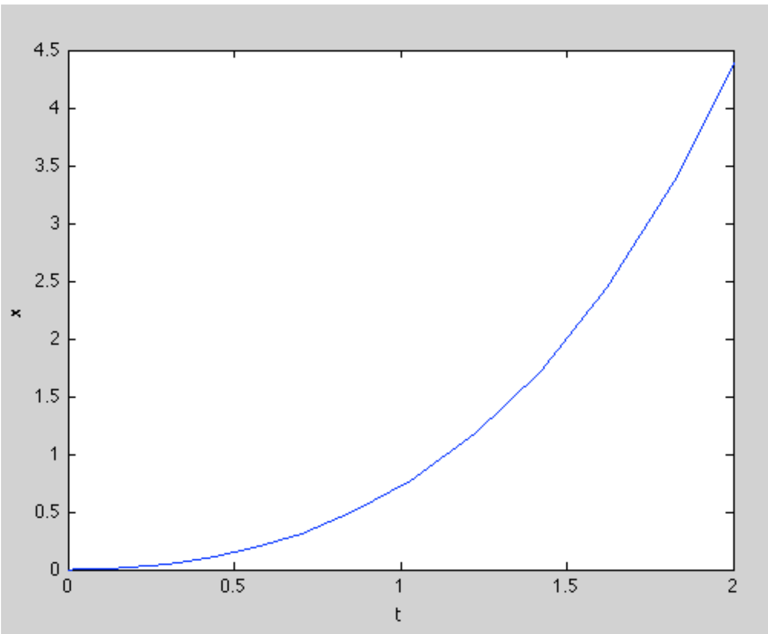
Step 2: Write a function to compute the new derivatives and save it as an M-File named `simpode.m`

```
function xdot = simpode(t,x);
%SIMPODE: Computes xdot = x + t
%Call syntax: xdot = simpode(t,x);
xdot = x + t;
```

Step 3: Use ode23 to compute solution in the command window

```
>>tspan = [0 2];          %Specify time span
>>x0 = 0;             %Specify x0
>>[t,x] = ode23('simpode',tspan,x0); % Now executes ode23
>>plot(t,x)           %plot t vs. x
>>xlabel('t')         %label x-axis
>>ylabel ('x')        %label y-axis
```

the plot generated by the above commands are shown below



Excercise 2. Solve the simple ODE with the initial condition x(0)=1

Step 1: Write the equation(s) as a system of first-order equations:

Step 2: Write a function to compute the new derivatives and save it as an M-File named `simpode1.m`

```
function xdot = simpode(t,x);
%SIMPODE: Computes xdot = x + t
%Call syntax: xdot = simpode(t,x);
xdot = -10*x;
```

Step 3: Use ode23 to compute solution in the command window

```
>>tspan = [0 1];     %Specify time span
>>x0 = 1;            %Specify x0
>>[t,x] = ode23('simpode1',tspan,x0); % Now executes ode23
>>plot(t,x)          %plot t vs. x
>>xlabel('t')        %label x-axis
>>ylabel ('x')       %label y-axis
```

the plot generated by the above commands are shown below